PROBLEM SET 5

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In the following exercises X is a locally compact Hausdorff space.

Problem 1. *If μ is a Radon measure and $f \in L^1(\mu)$, show that $\nu(E) = \int_E f d\mu$ is a Radon measure.*

Proof. We will need the following lemma.

Lemma (Corollary 3.6)**.** *If $f \in L^1(\mu)$, for every $\epsilon > 0$ there exists $\delta > 0$ so that $|\int_E f d\mu| < \epsilon$ whenever $\mu(E) < \delta$.*

Back to the problem, we need to show, by definition, ν is finite on compact sets, outer regular on Borel sets and inner regular on open sets. Let's fix an arbitrarily small $\epsilon > 0$ and thus have a δ as in the above lemma.

(1) (finite on compact sets) Let $K \subset X$ be a compact set, then
$$\nu(K) = \int_K f d\mu \leq \int_X f d\mu = \|f\|_{L^1} < \infty.$$
(2) (outer regular on Borel sets) Let $E \subset X$ be a Borel set and let U be an open set containing E with $\mu(U \backslash E) < \delta$, such U exists by outer regularity of μ. Then
$$\nu(U \backslash E) = \int_{U \backslash E} f d\mu < \epsilon.$$
This proves outer regularity of ν.
(3) (inner regularity on open sets) Let $U \subset X$ be an open set and K be a compact set contained in U with $\mu(U \backslash K) < \delta$, such K exists by inner regularity of μ. Then
$$\nu(U \backslash K) = \int_{U \backslash K} f d\mu < \epsilon.$$
This proves inner regularity of ν.

□

Problem 2. *If μ is a Radon measure and $f \in L^1(\mu)$ is real-valued, show that for every $\epsilon > 0$ there are an l.s.c. function g and a u.s.c. function h such that $h \leq f \leq g$ and $\int (g - h) d\mu < \epsilon$.*

Proof. If f is non-negative, then the result follows from Prop 7.14 and $\{x : f(x) > 0\}$ is σ-finite. To see this, we observe
$$\{x : f(x) > 0\} = \cup_n \{x : f(x) > 1/n\}$$
and $\mu(\{x : f(x) > 1/n\}) < n\|f\|_{L^1} < \infty$. Then by prop 7.14 one may find desired (upper and lower) semicontinuous functions $h \leq f \leq g$ such that
$$\int (g - f) d\mu < \epsilon/4, \quad \int (f - h) d\mu < \epsilon/4.$$

In general, we break f into its positive and negative parts $f = f^+ - f^-$ where both f^+, f^- are non-negative integrable functions. Then as before, we may find l.s.c. $g^\pm$ and u.s.c. $h^\pm$ with $h^\pm \leq f^\pm \leq g^\pm$ and

$$\int (g^\pm - f^\pm) d\mu < \epsilon/4, \quad \int (f^\pm - h^\pm) d\mu < \epsilon/4.$$

Let $h = h^+ - g^-$ and $g = g^+ - h^-$. Then h is u.s.c. since both h^+ and $-g^-$ are, similarly g is l.s.c. . Moreover $h \leq f \leq g$ and

$$\int (g-h) d\mu = \int (g^+ - h^+) d\mu + \int (g^- - h^-) d\mu < \epsilon.$$

□

Problem 3. *If μ is a positive Radon measure on X with $\mu(X) = \infty$, show that there exists $f \in C_0(X)$ such that $\int f d\mu = \infty$. Consequently, every positive linear functional on $C_0(X)$ is bounded.*

Proof. Since $\mu(X) = \infty$, one can find K_1 compact such that $\infty > \mu(K_1) \geq 1$. Then by outer regularity, one can find U_1 open with $K_1 \subset U_1$ and $\mu(U_1) \leq \mu(K_1) + 1$. Since X is LCH, we may further find V_1 open so that $K_1 \subset V_1 \subset \overline{V}_1 \subset U_1$. Note that all these sets have finite measure. Now we may inductively find a sequence of triples K_n, V_n, U_n, by replacing X by open set $X \setminus \cup_{1 \leq i \leq n-1} \overline{V}_i$ and notice $\mu(X \setminus \cup_{1 \leq i \leq n-1} \overline{V}_i) = \infty$, such that

(1) K_n compact, V_n, U_n open, $\overline{V_n}$ compact, $K_n \subset V_n \subset \overline{V}_n \subset U_n \subset X \setminus \cup_{1 \leq i \leq n-1} \overline{V}_i$. In particular, all V_n's are disjoint.
(2) $\mu(K_n) \geq 1$, $\mu(U_n) \leq \nu(K_n) + 1$.

Let $f_n \in C_c(X, [0,1])$ be a function with $f_n|_{K_n} = 1$ and $\operatorname{supp} f_n \subset V_n$, such function exists by Urysohn's lemma. Then $\sum_{n=1}^\infty f_n(x)$ trivially converges to a continuous function, call it f, since the f_n's have support on disjoint open sets V_n's. Moreover $f \in C_0(X)$ since for any $a > 0$, $\{x : |f(x)| \geq a\}$ is contained in union of finitely many compact sets $\overline{V}_n$'s, and

$$\int_X f d\mu \geq \sum_{n=1}^\infty \int_{K_n} f d\mu \geq \sum_{n=1}^\infty \mu(K_n) = \infty$$

as desired.

If I is a positive linear functional on $C_0(X)$, assume for contradiction that I is unbounded, that is

$$\sup\{I(f) \in C_0(X) : \|f\|_u \leq 1\} = \infty.$$

Then there exists $f_n \in C_0(X)$ with $\|f_n\|_u \leq 1$ so that $I(f_n) \geq 2^n$. Since $I(|f_n|) \geq I(f_n)$ by positivity of I, we may assume $f_n \geq 0$ for all n. Consider the series $\sum_{n=1}^\infty f_n/2^n$, it converges uniformly to a function $f \in C_0(X)$. Take $m \in \mathbb{N}$ big such that $m > I(f)$, then by positivity of I we have

$$m > I(f) \geq I(\sum_{n=1}^m f_n/2^n) \geq \sum_{n=1}^m I(f_n)/2^n \geq m$$

which is a contradiction.

□

Problem 4. *If μ is a σ-finite Radon measure on X and $\nu \in M(X)$, let $\nu = \nu_1 + \nu_2$ be the Lebesgue decomposition of ν with respect to μ. Show that ν_1 and ν_2 are Radon.*

Proof. We may assume $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$. Then by Lebesgue-Radon-Nikodym theorem, $\nu_1 = f d\mu$ for some $f \in L^1(\mu)$, hence ν_1 is Radon by Problem 1, that is $\nu_1 \in M(x)$. Since $M(X)$ is a complex vector space, $\nu_2 = \nu - \nu_1 \in M(X)$, *i.e.* ν_2 is Radon. □

Problem 5. *Show that a sequence $\{f_n\}$ in $C_0(X)$ converges weakly to $f \in C_0(X)$ iff $\sup_n \|f_n\|_u < \infty$ and $f_n \to f$ pointwise.*

Proof. If f_n converges weakly to $f \in C_0(X)$, then $\sup_n \|f_n\|_u < \infty$ follows from HW3, Problem 5. To see f_n converges to f pointwise, for each $x \in X$, we consider mass measure δ_x, then $f_n(x) = \int_X f_n d\delta_x \to \int_X f d\delta_x = f(x)$. Conversely, if $\sup_n \|f_n\|_u < \infty$ and f_n converges to f pointwise, consider $g(x) = \sup_n |f_n(x)|$. We have $\|g\|_u \leq \sup_n \|f_n\| < \infty$, therefore $|\int_X |g| d\mu| \leq \|g\|_u \|\mu\| < \infty$ and hence $g \in L^1(\mu)$ for all $\mu \in M(X)$. Notice $\{f_n\}$ is dominated by g, so by dominated convergence theorem, $\int_X f_n d\mu \to \int_X f d\mu$ for all $\mu \in M(X)$. □

Problem 6. *Find examples of sequences $\{\mu_n\}$ in $M(\mathbb{R})$ such that:*

(1) $\mu_n \to 0$ *vaguely, but* $\|\mu_n\| \not\to 0$.
(2) $\mu_n \to 0$ *vaguely, but* $\int f d\mu_n \not\to \int f d\mu$ *for some bounded measurable* f *with compact support.*
(3) $\mu_n \geq 0$ *and* $\mu_n \to \mu$ *vaguely, but, there exists* $x \in \mathbb{R}$ *such that* $F_n(x) \not\to F(x)$.

Proof.
(1) Let $\mu_n = \chi_{[-n-1,-n]} dx$ where dx is the Lebesgue measure. Then for any $f \in C_0(X)$ we may find K compact outside which $|f|$ is smaller than a given ϵ, thus for n big enough $|\int_{-n-1}^{-n} f dx| < \epsilon$, this proves $\mu_n \to 0$ vaguely. But apparently $\|\mu_n\| = 1$ for all n.
(2) $\delta_{1/n}$ converges vaguely to δ_0 since $f(1/n) \to f(0)$ for all $f \in C_0(X)$. But $\int \chi_{\{0\}} d\delta_{1/n} = 0$ does not converge to $\int \chi_{\{0\}} d\delta_0 = 1$.
(3) The example in (1) also provides an example for (3), simply observe $F_n(0) = 1$ for all n but $F(0) = 0$.

□

Problem 7. *Let μ be a Radon measure on X (which we assume is first countable) such that every open set has positive measure. Show that for each $x \in X$ there is a sequence $\{f_n\}$ in $L^1(\mu)$ which converges vaguely in $M(X)$ to the point mass at x.*

Proof. Let $\{U_n\}$ be a decreasing neighborhood basis of x, let $f_n = \chi_{U_n}/\mu(U_n)$ which is trivially L^1. We claim $f_n d\mu \to \delta_x$ vaguely. Indeed for any $f \in C_0(X)$, we may find open neighborhood $x \in V$ so that $|f(y) - f(x)| < \epsilon$ for a fixed ϵ, then

$$\left|\frac{1}{\mu(U_n)} \int_{U_n} f(y) d\mu(y) - f(x)\right| \leq \epsilon$$

for n big enough so that $U_n \subset V$. □